

LACUNARY FOURIER AND WALSH-FOURIER SERIES NEAR L^1

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ABSTRACT. We prove the following theorem: given a lacunary sequence of integers $\{n_j\}$, the subsequences $F_{n_j}f$ and $W_{n_j}f$ of respectively the Fourier and the Walsh-Fourier series of $f : \mathbb{T} \rightarrow \mathbb{C}$ converge almost everywhere to f whenever

$$(1) \quad \int_{\mathbb{T}} |f(x)| \log \log(e^e + |f(x)|) \log \log \log \log(e^{eee} + |f(x)|) dx < \infty.$$

Our integrability condition (1) is less stringent than the homologous assumption in the almost everywhere convergence theorems of Lie [13] (Fourier case) and Do-Lacey [6] (Walsh-Fourier case), where a triple-log term appears in place of the quadruple-log term of (1). Our proof of the Walsh-Fourier case is self-contained and, in antithesis to [6], avoids the use of Antonov's lemma [1], arguing directly via weak- L^p bounds for the Walsh-Carleson operator.

1. INTRODUCTION AND MAIN RESULT

Let $\mathbb{T} : \mathbb{R} \setminus \mathbb{Z}$ be the one dimensional torus, identified with the interval $[0, 1)$, and write

$$\langle f, g \rangle = \int_{\mathbb{T}} f(x) \overline{g(x)} dx.$$

For each $f \in L^1(\mathbb{T})$, one can construct the Fourier series of f

$$F_n f(x) = \sum_{k=-n}^n \langle f, E_k \rangle E_k(x), \quad x \in \mathbb{T}, n \in \mathbb{N}$$

where $E_k(x) = e^{2\pi i k x}$, as well as the Walsh-Fourier series of f

$$W_n f(x) = \sum_{k=0}^n \langle f, W_k \rangle W_k(x), \quad x \in \mathbb{T}, n \in \mathbb{N}$$

where $\{W_n : n \in \mathbb{N}\}$ is the orthonormal basis of $L^2(\mathbb{T})$ defined as

$$W_n(x) = \prod_{k \in \mathbb{N}} (\text{sign} \sin(2^k 2\pi x))^{\varepsilon_k(n)}, \quad \varepsilon_k(n) := \lfloor 2^{-k} n \rfloor \bmod 2.$$

We are interested in almost-everywhere convergence of $F_n f$, $W_n f$ along lacunary subsequences of integers $\{n_j : j \in \mathbb{N}\}$, that is, sequences of integers for which

$$\inf_{j \in \mathbb{N}} \frac{n_{j+1}}{n_j} = \theta > 1;$$

the constant θ is termed the *lacunarity constant* of the sequence $\{n_j\}$.

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The aim of this note is to prove Theorem 1 below. In the statement, as well as in the remainder of the paper, we adopt the notations

$$\log_k(t) = \underbrace{\log(\cdots(\log(e_k + t)\cdots))}_{k \text{ times}}, \quad e_0 = 1, \quad e_k := e^{e_{k-1}}, \quad k = 0, 1, 2, \dots$$

The precise definition of the Orlicz spaces $L\log_2 L\log_b L(\mathbb{T})$, $b = 3, 4$, appearing in the statement of the theorem and in the subsequent discussion is postponed at the end of the introduction.

Theorem 1. *Let $\mathbf{n} = \{n_j : j \in \mathbb{N}\}$ be a θ -lacunary sequence of integers. The lacunary Carleson (resp. Walsh-Carleson) maximal operators*

$$F_{\mathbf{n}}^* f(x) := \sup_{n \in \mathbf{n}} |F_n f(x)|, \quad W_{\mathbf{n}}^* f(x) := \sup_{n \in \mathbf{n}} |W_n f(x)|$$

map $L\log_2 L\log_4 L(\mathbb{T})$ into $L^{1,\infty}(\mathbb{T})$, with operator norms depending only on θ . As a consequence, almost everywhere convergence of the lacunary partial sums

$$F_{n_j} f(x) \rightarrow f(x), \quad W_{n_j} f(x) \rightarrow f(x), \quad \text{a.e. } x \in \mathbb{T}$$

holds for all $f \in L\log_2 L\log_4 L(\mathbb{T})$.

We send the interested reader to the survey article [11] and references therein for additional context and perspective on problems related to the almost-everywhere convergence of Walsh and of Fourier series (in particular, along lacunary subsequences). Here, we mention that Theorem 1 without the \log_4 term, which is the object of a conjecture by Konyagin [11], would be sharp in the following sense: for any nondecreasing $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ and

$$\phi(t) = o(t \log_2(t)), \quad t \rightarrow \infty,$$

and any lacunary sequence $\{n_j\}$ there exists a function f in $\phi(L)$ with lacunary Fourier series divergent everywhere, i.e.

$$\int_{\mathbb{T}} \phi(|f(x)|) dx < \infty \quad \text{and} \quad \sup_j |F_{n_j} f(x)| = \infty \quad \forall x \in \mathbb{T}.$$

This is due to Konyagin [10] as well; a perusal of the proof extends the construction to the Walsh-Fourier case.

The recent articles [6] and [13] have made significant progress towards a positive solution of Konyagin's conjecture, respectively in the Walsh and in the Fourier setting. Their respective main results can be summarized as follows: given any lacunary sequence of integers $\{n_j\}$, the subsequence $W_{n_j} f$ [resp. $F_{n_j} f$] converges almost everywhere to f for all $f \in L\log_2 L\log_3 L(\mathbb{T})$.

The bulk of [6] is devoted to the proof of the following restricted weak-type estimate for the lacunary Walsh-Carleson maximal operator: for all θ -lacunary sequences \mathbf{n} ,

$$(1.1) \quad \|W_{\mathbf{n}}^* f\|_{1,\infty} \leq K |F| \log_2 \left(\frac{1}{|F|} \right), \quad \forall |f| \leq \mathbf{1}_F, F \subset \mathbb{T},$$

where K is a positive constant depending only on the lacunarity constant θ of \mathbf{n} . A subsequent application of Antonov's lemma [1] improves (1.1) into the (modified) weak-type estimate

$$(1.2) \quad \|W_{\mathbf{n}}^* f\|_{1,\infty} \leq K \|f\|_1 \log_2 \left(\frac{\|f\|_{\infty}}{\|f\|_1} \right)$$

for all bounded functions $f : \mathbb{T} \rightarrow \mathbb{C}$. In the later article [13], a direct (that is, without first proving a restricted weak-type estimate and then achieving weak-type via Antonov's lemma) proof of the Fourier analogue of (1.2), namely

$$(1.3) \quad \|F_{\mathbf{n}}^* f\|_{1,\infty} \leq K \|f\|_1 \log_2 \left(\frac{\|f\|_\infty}{\|f\|_1} \right).$$

is given. Once estimates (1.2)-(1.3) are in place, the bounds

$$(1.4) \quad W_{\mathbf{n}}^*, F_{\mathbf{n}}^* : \mathfrak{W} \rightarrow L^{1,\infty}(\mathbb{T}),$$

the space \mathfrak{W} being the quasi-Banach rearrangement invariant space with quasinorm

$$(1.5) \quad \|f\|_{\mathfrak{W}} := \inf \left\{ \sum_{k \in \mathbb{N}} \log_1(k) \|f_k\|_1 \log_2 \left(\frac{\|f_k\|_\infty}{\|f_k\|_1} \right) : \begin{array}{l} f = \sum_{k \in \mathbb{N}} f_k, \\ \sum_{k \in \mathbb{N}} |f_k| < \infty \text{ a.e.} \end{array} \right\},$$

follow, as described in [13], from an exploitation of Kalton's log-convexity of $L^{1,\infty}(\mathbb{T})$ [8]. A standard density argument then implies almost everywhere convergence of $W_{\mathbf{n}}^* f, F_{\mathbf{n}}^* f$ for functions $f \in \mathfrak{W}$. The space \mathfrak{W} is akin to the QA space of [2], and the embedding

$$(1.6) \quad L \log_2 L \log_3 L(\mathbb{T}) \hookrightarrow \mathfrak{W}$$

follows along the lines of the theory developed in [2] for QA . In view of the above discussion, coupling the embedding (1.6) with (1.4) immediately leads to the main results of respectively [6] and [13].¹ Our observation is that, in fact, the strengthening of (1.6)

$$(1.7) \quad L \log_2 L \log_4 L(\mathbb{T}) \hookrightarrow \mathfrak{W}$$

holds as well, whence, assuming (1.4) again (for e. g. in the Walsh case)

$$(1.8) \quad \|W_{\mathbf{n}}^* f\|_{1,\infty} \leq K \|f\|_{L \log_2 L \log_4 L(\mathbb{T})},$$

which in turn implies the almost everywhere convergence part of Theorem 1. The (elementary) proof of (1.7) is given in Section 2.

We also give a self-contained proof of the inequality (1.2), and hence of the Walsh-Fourier case of the bound (1.4). Our proof is both simpler, and richer, than the one of [6]: in particular, in antithesis to [6], we bypass the intermediate step (1.1), thus avoiding the need for Antonov's lemma. We argue instead via the weak-type bound of Theorem 2 below, which possesses intrinsic interest.

Theorem 2. *Let $\mathbf{n} = \{n_j\}$ be a θ -lacunary sequence. There is a positive constant K , depending only on the lacunarity constant θ of \mathbf{n} , such that, for all $1 < p \leq 2$*

$$\|W_{\mathbf{n}}^* f\|_{p,\infty} \leq K \log_1(p') \|f\|_p.$$

Indeed, with Theorem 2 in hand, (1.2) follows easily via the chain of inequalities

$$\|W_{\mathbf{n}}^* f\|_{1,\infty} \leq \inf_{p>1} \|W_{\mathbf{n}}^* f\|_{p,\infty} \leq \|W_{\mathbf{n}}^* f\|_{\bar{p},\infty} \leq K \log_1(\bar{p}') \|f\|_{\bar{p}} \leq K \log_1(\bar{p}') \|f\|_1 \left(\frac{\|f\|_\infty}{\|f\|_1} \right)^{\frac{1}{\bar{p}'}}$$

¹The authors of [6] employ a differently defined quasi-Banach space, denoted Q_D , and derive $Q_D \rightarrow L^{1,\infty}(\mathbb{T})$ boundedness of $W_{\mathbf{n}}^*$ from (1.2), as well as the embedding $L \log_2 L \log_3 L(\mathbb{T}) \hookrightarrow Q_D$, by appealing to the results of [3], which generalize Arias De Reyna's work [2]. However, it can be inferred from the discussion in [3, Section 1] that the spaces Q_D and \mathfrak{W} coincide in this particular case.

finally taking $\bar{p}' = \max \{2, \log(\frac{\|f\|_\infty}{\|f\|_1})\}$.

A more detailed comparison of our approach to the proof in [6], and a discussion on sharpness of Theorem 2, are given in the remarks Section 5. Here, we just mention that one of the main tools of our proof (appearing, albeit in a different form, in [6] as well) is a lacunary multifrequency Calderón-Zygmund decomposition argument (here, Lemma 4.1), along the lines of [15, Theorem 1.1]. The structural² obstruction to this scheme of proof when dealing with the Fourier case is that the mean zero (with respect to multiple frequencies) part arising from the multifrequency CZ decomposition, informally known as “the bad part”, brings nontrivial contribution, unlike the Walsh case. Despite the additional cancellation, we are unable to estimate this contribution efficiently as of now: overcoming these difficulties will be the object of future work.

Notation. We will indicate by \mathcal{D} the standard dyadic grid on $\mathbb{R}_+ = [0, \infty)$ and by $\mathcal{D}_I = \{J \in \mathcal{D} : J \subseteq I\}$. Throughout, given a Young’s function φ , we make use of the local Orlicz norms

$$\|f\|_{L^\varphi(I)} := \inf \left\{ \lambda > 0 : \int_I \varphi\left(\frac{|f(x)|}{\lambda}\right) \frac{dx}{|I|} \leq 1 \right\}, \quad I \in \mathcal{D}.$$

When $\varphi(t) = t^p$, $1 \leq p \leq \infty$, we simply write $L^p(I)$. With this notation, the usual L^p Hardy-Littlewood dyadic maximal function is defined by

$$M_p f(x) = \sup_{\mathcal{D} \ni I \ni x} \|f\|_{L^p(I)}.$$

With the notation $L \log_2 L \log_b L(\mathbb{T})$, $b = 3, 4$ we indicate the Orlicz (Banach) space defined by any Young’s function φ_b with $t \log_2(t) \log_b(t) = \varphi_b(t)$ for $t > e_b$. We observe for future use that $L \log_2 L \log_b L(\mathbb{T})$ is a Banach space with unit ball

$$B_b = \left\{ f : \mathbb{T} \rightarrow \mathbb{C}, |f|_{\mathcal{L} \log_2 \mathcal{L} \log_b \mathcal{L}} := \int_{\mathbb{T}} \varphi_b(|f(x)|) dx \leq 1 \right\},$$

Finally, the positive constants implied by the almost inequality signs appearing in the remainder of the paper are meant to be absolute unless otherwise specified: in that case, we will adopt the notation \lesssim_a to indicate dependence of the implied constant on the parameter a . When we write $A \sim B$, we mean that $A \lesssim B$ and $B \lesssim A$ (and analogously for \sim_a).

Plan of the paper. In the forthcoming Section 2, we prove (1.7), which in turn implies Theorem 1, via estimates (1.2), (1.3). In Section 3, we review the discretization of the operator W_n^* into the model sum C^n and prove an auxiliary exponential estimate. This exponential estimate, together with a multi-frequency projection argument exploiting the lacunary structure of the frequencies (Lemma 4.1), are the cornerstones of the proof of Theorem 2, given in Section 4. Section 5 contains additional remarks and open problems.

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²That is, modulo the usual technicalities due to the spatial tails of the Fourier wave packets.

2. PROOF OF THE EMBEDDING (1.7)

To prove (1.7), in view of the definition (1.5) of the quasinorm on \mathfrak{M} , it suffices to show that for any f in the unit ball B_4 of $L \log_2 L \log_4 L(\mathbb{T})$ there exists a sequence $\{f_k : k \in \mathbb{N}\}$ with

$$(2.1) \quad f = \sum_{k \in \mathbb{N}} f_k, \quad \sum_{k \in \mathbb{N}} |f_k| < \infty \text{ a.e.}, \quad \sum_{k \in \mathbb{N}} \log_1(k) \|f_k\|_1 \log_2 \left(\frac{\|f_k\|_\infty}{\|f_k\|_1} \right) \lesssim 1.$$

Given such an $f \in B_4$, we define $\{f_k : k \in \mathbb{N}\}$ by

$$f_k := f \mathbf{1}_{F_k}, \quad F_0 = \{|f| \leq e^e\}, \quad F_k = \left\{ e^{e^{e^k}} < |f| \leq e^{e^{e^{k+1}}} \right\}, \quad k \geq 1.$$

The absolute convergence almost everywhere of the series is immediate, since each f_k is bounded and the supports of the $|f_k|$ are pairwise disjoint. We use the elementary fact that

$$x \in F_k \implies \log_2(|f(x)|) \log_4(|f(x)|) \sim e^k \log_1 k.$$

Consequently, adopting the shorthand $A_k := |f_k|_{\mathcal{L} \log_2 \mathcal{L} \log_4 \mathcal{L}}$

$$\|f_k\|_1 \sim \frac{A_k}{e^k \log_1(k)}, \quad \frac{\|f_k\|_\infty}{\|f_k\|_1} \lesssim \frac{e^{e^{e^{k+1}}} e^k \log_1(k)}{A_k},$$

whence

$$(2.2) \quad \|f_k\|_1 \log_2 \left(\frac{\|f_k\|_\infty}{\|f_k\|_1} \right) \lesssim \frac{A_k}{e^k \log_1(k)} \log_2 \left(\frac{e^{e^{e^{k+1}}} e^k \log_1(k)}{A_k} \right).$$

We separate two regimes. In the regime

$$R_1 = \left\{ k : \frac{A_k}{e^k \log_1(k)} \geq \frac{1}{e^{e^{e^{k+1}}}} \right\},$$

the above inequality turns into

$$(2.3) \quad \|f_k\|_1 \log_2 \left(\frac{\|f_k\|_\infty}{\|f_k\|_1} \right) \lesssim \frac{A_k}{e^k \log_1(k)} \log_2 \left((e^{e^{e^{k+1}}})^2 \right) \lesssim \frac{A_k}{\log_1(k)}.$$

In the complementary regime R_2 , using the trivial inequalities

$$4 \log_2(ab) \leq 2 \log_2(a) \log_2(b), \quad \forall a, b > 0$$

and $a \log \frac{1}{a} \leq \sqrt{a}$ for $|a| \leq 1$, (2.2) becomes

$$(2.4) \quad \begin{aligned} \|f_k\|_1 \log_2 \left(\frac{\|f_k\|_\infty}{\|f_k\|_1} \right) &\lesssim \frac{A_k}{e^k \log_1(k)} \log_2 \left(\frac{e^k \log_1(k)}{A_k} \right) \log_2 \left(e^{e^{e^{k+1}}} \right) \\ &\lesssim \left(\frac{A_k}{e^k \log_1(k)} \right)^{\frac{1}{2}} e^k \lesssim \frac{e^k}{e^{e^{\frac{k}{2}}}} \lesssim e^{-k}. \end{aligned}$$

With (2.3)-(2.4) in hand, we easily get the last part of (2.1) as follows:

$$\sum_{k \in \mathbb{N}} \log_1(k) \|f_k\|_1 \log_2 \left(\frac{\|f_k\|_\infty}{\|f_k\|_1} \right) \lesssim \sum_{k \in R_1} A_k + \sum_{k \in R_2} e^{-k} \log_1(k) \lesssim |f|_{\mathcal{L} \log_2 \mathcal{L} \log_4 \mathcal{L}} + 1 \lesssim 1.$$

The proof of the embedding (1.7) is therefore complete.

3. DISCRETIZATION AND AN EXPONENTIAL ESTIMATE

A *bitile* $s = I_s \times \omega_s \in \mathcal{D}_{\mathbb{T}} \times \mathcal{D}$ is a dyadic rectangle with $|\omega_s| = 2|I_s|^{-1}$. We think of s as the union of the two *tiles* (dyadic rectangles in $\mathcal{D}_{\mathbb{T}} \times \mathcal{D}$ of area 1)

$$s_1 = I_s \times \omega_{s_1}, \quad s_2 = I_s \times \omega_{s_2}$$

where $\omega_{s_1}, \omega_{s_2}$ refer respectively to the left and right dyadic children of ω_s . The set of all bitiles will be denoted by $\mathbf{S}_{\mathbb{T}}$. For each tile $t = I_t \times \omega_t$, the corresponding Walsh wave packet is defined by

$$w_t(x) = \text{Dil}_{|I_t|}^2 \text{Tr}_{\inf I_t} W_{n_t}(x) = |I_t|^{-1/2} W_{n_t}\left(\frac{x - \inf I_t}{|I_t|}\right), \quad n_t := |I_t| \inf \omega_t.$$

Let $N : \mathbb{T} \rightarrow \mathbb{R}_+$ be a measurable choice function and consider the model sum

$$C_{\mathbf{S}_{\mathbb{T}}} f(x) = \sum_{s \in \mathbf{S}_{\mathbb{T}}} \langle f, w_{s_1} \rangle w_{s_1}(x) \mathbf{1}_{\omega_{s_2}}(N(x));$$

we do not indicate the dependence on the choice function in our notation. This model sum is the discretization of the *unrestricted* maximal operator $W^* f := \sup_{n \in \mathbb{N}} |W_n f|$. To obtain a faithful model sum for the maximal partial sum $W_{\mathbf{n}}^*$ restricted to the (lacunary) sequence $\mathbf{n} = \{n_j\}$, we restrict the range of the choice function N to values in \mathbf{n} ; this restricts the sum over the bitiles $\mathbf{S}_{\mathbb{T}}^{\mathbf{n}} := \{s \in \mathbf{S}_{\mathbb{T}} : \omega_{s_2} \cap \mathbf{n} \neq \emptyset\}$, whence the equivalence [18] $W_{\mathbf{n}}^* f \sim C_{\mathbf{S}_{\mathbb{T}}^{\mathbf{n}}} f$. In the remainder of the article, we use the simpler notation $C^{\mathbf{n}}$ in place of $C_{\mathbf{S}_{\mathbb{T}}^{\mathbf{n}}}$ and further denote by

$$C_{\mathbf{S}} f(x) = \sum_{s \in \mathbf{S}} \langle f, w_{s_1} \rangle w_{s_1}(x) \mathbf{1}_{\omega_{s_2}}(N(x))$$

the model sum corresponding to an arbitrary finite subcollection $\mathbf{S} \subset \mathbf{S}_{\mathbb{T}}^{\mathbf{n}}$.

The remainder of this section is devoted to the following proposition, on whose proof Theorem 2 relies upon.

Proposition 3.1. *Let $\mathbf{n} = \{n_j\}$ be a θ -lacunary sequence. Then $C^{\mathbf{n}} : L^\infty(\mathbb{T}) \rightarrow \exp(L^1)(\mathbb{T})$, that is*

$$|\{x \in \mathbb{T} : |C^{\mathbf{n}} f(x)| \gtrsim \lambda\}| \lesssim_{\theta} \exp\left(-\frac{\lambda}{\|f\|_{\infty}}\right), \quad \lambda > 0.$$

The proof of Proposition 3.1 is given in Subsection 3.2; in the forthcoming Subsection 3.1, we recall the necessary tools of time-frequency analysis.

3.1. Analysis and combinatorics in the Walsh phase plane. The material of this subsection is essentially lifted from earlier work [5] (see also [18]), with the exception of Lemma 3.4, which exploits the lacunary structure of the frequencies.

We begin by recalling the well-known Fefferman order relation on either tiles or bitiles

$$(3.1) \quad s \ll s' \iff I_s \subset I_{s'} \text{ and } \omega_s \supset \omega_{s'}.$$

A collection $\mathbf{S} \subset \mathbf{S}_{\mathbb{T}}$ is called *convex* if

$$(3.2) \quad s, s'' \in \mathbf{S}, s' \in \mathbf{S}_{\mathbb{T}}, s \ll s' \ll s'' \implies s' \in \mathbf{S}.$$

We will use below that the collection of convex subsets is closed under finite intersection.

Given a set of bitiles \mathbf{S} , let $\Pi_{\mathbf{S}}$ denote the orthogonal projection on the subspace of $L^2(\mathbb{T})$ spanned by $\{w_{s_j} : s \in \mathbf{S}, j = 1, 2\}$. We set, for $f \in L^2(\mathbb{T})$,

$$\text{size}_f(\mathbf{S}) = \sup_{s \in \mathbf{S}} \frac{\|\Pi_{\{s\}} f\|_2}{\sqrt{|I_s|}}.$$

Note that

$$\text{size}_f(\mathbf{S}) \sim \sup_{s \in \mathbf{S}} \sup_{j=1,2} \frac{|\langle f, w_{s_j} \rangle|}{\sqrt{|I_s|}}.$$

so that

$$(3.3) \quad \text{size}_f(\mathbf{S}) \leq \sup_{s \in \mathbf{S}} \inf_{x \in I_s} M_1 f(x).$$

A collection of bitiles $\mathbf{T} \subset \mathbf{S}$ is called *tree* with top bitile $s_{\mathbf{T}}$ if $s \ll s_{\mathbf{T}}$ for all $s \in \mathbf{T}$. We use the notation $I_{\mathbf{T}} := I_{s_{\mathbf{T}}}$, $\omega_{\mathbf{T}} = \omega_{s_{\mathbf{T}}}$. To characterize the contribution region of a tree, it is useful to introduce the notion of *crown* of a tree:

$$\text{cr}(\mathbf{T}) = \bigcup_{s \in \mathbf{T}} \omega_{s_2}.$$

We have the following exponential-type estimate for the model sum restricted to a tree of definite size. Note that $C_{\mathbf{T}}f$ is supported on $I_{\mathbf{T}}$.

Lemma 3.2. *Let \mathbf{T} be a convex tree and $\sigma = \text{size}_f(\mathbf{T})$. Then*

$$|\{x \in I_{\mathbf{T}} : |C_{\mathbf{T}}f(x)| \gtrsim \lambda \sigma\}| \lesssim e^{-\lambda} |I_{\mathbf{T}}|, \quad \forall \lambda > 0.$$

Proof. It is obvious that $C_{\mathbf{T}}f = C_{\mathbf{T}}\Pi_{\mathbf{T}}f$, hence the lemma follows from the bound

$$\|C_{\mathbf{T}}(\Pi_{\mathbf{T}}f)\|_{\text{BMO}(\mathbb{T})} \lesssim \|\Pi_{\mathbf{T}}f\|_{\infty} \leq \text{size}_f(\mathbf{T})$$

and John-Nirenberg inequality. For details on the second inequality see (for instance) [5]. \square

A finite convex collection of bitiles \mathbf{S} is called a *forest* if \mathbf{S} can be partitioned into (pairwise disjoint) convex trees $\{\mathbf{T} : \mathbf{T} \in \mathcal{F}\}$. It may be that a given \mathbf{S} may admit many such partitions \mathcal{F} . The *counting* function and the *crown* function of the forest \mathbf{S} with respect to the partition \mathcal{F} are respectively defined as

$$\mathcal{N}_{\mathcal{F}}(x) = \sum_{\mathbf{T} \in \mathcal{F}} \mathbf{1}_{I_{\mathbf{T}}}(x), \quad \mathcal{W}_{\mathcal{F}}(x) = \sum_{\mathbf{T} \in \mathcal{F}} \mathbf{1}_{I_{\mathbf{T}}}(x) \mathbf{1}_{\text{cr}(\mathbf{T})}(N(x))$$

For a tree \mathbf{T} , $\text{supp } C_{\mathbf{T}}f \subset I_{\mathbf{T}} \cap N^{-1}(\text{cr}(\mathbf{T}))$, and as a consequence, for a forest \mathbf{S} with partition \mathcal{F} , one has the pointwise inequality

$$(3.4) \quad |C_{\mathbf{S}}f(x)| \leq \mathcal{W}_{\mathcal{F}}(x) \max_{\mathbf{T} \in \mathcal{F}} |C_{\mathbf{T}}f(x)|.$$

The lemma below can be used to decompose any convex collection of bitiles into forests of definite size, keeping the L^1 norm of the counting functions under control. See [5] for a proof.

Lemma 3.3. *Let \mathbf{S} be a finite convex collection of bitiles with $\text{size}_f(\mathbf{S}) \leq A$. We can decompose $\mathbf{S} = \bigcup \{\mathbf{S}_\sigma : \sigma \in 2^{-\mathbb{N}}\}$, with each \mathbf{S}_σ a forest such that*

$$(3.5) \quad \text{size}_f(\mathbf{S}_\sigma) \leq A\sigma,$$

$$(3.6) \quad \|\mathcal{N}_{\mathcal{F}_\sigma}\|_1 \lesssim \sigma^{-2} A^{-2} \|f\|_2^2,$$

for some partition \mathcal{F}_σ .

Our last lemma is specific of the lacunary case: in view of the fact that each bitile contains elements from the lacunary sequence \mathbf{n} , we have a bound on the crown function of a generic forest which only depends on the lacunarity constant θ .

Lemma 3.4. *For any forest $\mathbf{S} \subset \mathbf{S}_{\mathbb{T}}^{\mathbf{n}}$ with partition \mathcal{F} , there is a partition \mathcal{F}^* with*

$$\|\mathcal{W}_{\mathcal{F}^*}\|_\infty \lesssim_\theta 1, \quad \|\mathcal{N}_{\mathcal{F}^*}\|_1 \lesssim_\theta \|\mathcal{N}_{\mathcal{F}}\|_1.$$

Proof. It suffices to show that \mathbf{S} can be split into $\sim_\theta 1$ forests \mathbf{S}^j with partitions \mathcal{F}^j , such that

$$\|\mathcal{N}_{\mathcal{F}^j}\|_1 \lesssim_\theta \|\mathcal{N}_{\mathcal{F}}\|_1, \quad \{I_{\mathbf{T}} \times \text{cr}(\mathbf{T}) : \mathbf{T} \in \mathcal{F}^j\} \text{ pairwise disjoint.}$$

We define $\mathbf{S}^0 := \{s \in \mathbf{S} : n_1 \in \omega_s\}$. It is clear that \mathbf{S}^0 can be partitioned into convex trees $\mathbf{T} \in \mathcal{F}^0$ with pairwise disjoint $I_{\mathbf{T}}$ (take the \ll -maximal bitiles in \mathbf{S}^0 as tops). For each of these trees there exists a unique tree $\mathbf{T}' \in \mathcal{F}$ such that the top bitile $s_{\mathbf{T}} \in \mathbf{T}'$, whence $|I_{\mathbf{T}}| \leq |I_{\mathbf{T}}'|$; it then follows that $\|\mathcal{N}_{\mathcal{F}^0}\|_1 \leq \|\mathcal{N}_{\mathcal{F}}\|_1$. Let now $\tilde{\mathbf{S}} = \mathbf{S} \setminus \mathbf{S}^0$ and $\tilde{\mathbf{S}}^*$ be the \ll -maximal bitiles of $\tilde{\mathbf{S}}$. It should be apparent that $\sum_{s \in \tilde{\mathbf{S}}^*} |I_s| \leq \|\mathcal{N}_{\mathcal{F}}\|_1$. By the Fefferman trick (see for example Section 5 of [4]), the initial claim will follow if we show that for each $s \in \tilde{\mathbf{S}}$

$$M := \max_{s \in \tilde{\mathbf{S}}} \#(T(s) := \{s' \in \tilde{\mathbf{S}}^* : I_s \subset I_{s'}, \omega_{s'} \subset \omega_{s_2}\}) \lesssim_\theta 1$$

Take $s \in \tilde{\mathbf{S}}$ which attains the maximum M . The collection $T(s)$ is made of pairwise disjoint bitiles with $I_s \subset I_{s'}$, thus the intervals $\{\omega_s : s \in T(s)\}$ must be pairwise disjoint, and each contains a different $n_j \in \mathbf{n}$. It follows that ω_{s_2} contains at least M different frequencies. Let n_j and n_k be the minimum and the maximum of these frequencies respectively. It must be $k \geq j + M$, whence $|\omega_{s_2}| \geq n_k - n_j \geq (\theta^M - 1)n_j$. If $M \geq \frac{\log 2}{\log \theta}$, we would have $|\omega_{s_1}| \geq n_j$, $\inf \omega_{s_1} \leq n_j$, which in turn would imply $n_1 \in \omega_s$, and s would have been selected for \mathbf{S}^0 . Thus $M \leq \frac{\log 2}{\log \theta} \lesssim_\theta 1$ as claimed. \square

3.2. Proof of Proposition 3.1. It suffices to argue for $\lambda > \|f\|_\infty$ (the statement is otherwise trivial). Furthermore, by a limiting argument, we may argue for $C_{\mathbf{S}}$ in place of $C^{\mathbf{n}}$, with \mathbf{S} arbitrary finite convex subcollection of $\mathbf{S}_{\mathbb{T}}^{\mathbf{n}}$, ensuring that the implied constants do not depend on \mathbf{S} .

A consequence of (3.3) is that $\text{size}_f(\mathbf{S}) \leq \|f\|_\infty$, and we can apply the size decomposition Lemma 3.3, with $A = \|f\|_\infty$. We further apply Lemma 3.4 to the resulting forests $\{\mathbf{S}_\sigma\}_{\sigma \in 2^{-\mathbb{N}}}$ with $\text{size}_f(\mathbf{S}_\sigma) \leq \sigma \|f\|_\infty$, yielding partitions \mathcal{F}_σ with

$$(3.7) \quad \|\mathcal{N}_{\mathcal{F}_\sigma}\|_1 \lesssim \sigma^{-2} \|f\|_\infty^{-2} \|f\|_2^2, \quad \|\mathcal{W}_{\mathcal{F}_\sigma}\|_\infty \lesssim 1.$$

We will show that

$$(3.8) \quad \{|C_{\mathbf{S}} f| \gtrsim \lambda\} \subset E := \bigcup_{\sigma \in 2^{-\mathbb{N}}} \bigcup_{\mathbf{T} \in \mathcal{F}_\sigma} \left(E_{\mathbf{T}} := \{x \in I_{\mathbf{T}} : |C_{\mathbf{T}} f| \gtrsim \lambda \sigma \log\left(\frac{1}{\sigma^4}\right)\} \right);$$

note that, applying Lemma 3.2,

$$|E_T = \{x \in I_T : |C_{\mathbf{T}}f| \gtrsim \frac{\lambda}{\|f\|_\infty} \log\left(\frac{1}{\sigma^4}\right) \text{size}_f(\mathbf{T})\}| \lesssim \exp\left(-\frac{\lambda}{\|f\|_\infty}\right) \sigma^4 |I_{\mathbf{T}}|,$$

whence, in view of (3.7),

$$|E| \leq \exp\left(-\frac{\lambda}{\|f\|_\infty}\right) \sum_{\sigma \in 2^{-\mathbb{N}}} \sigma^4 \|\mathcal{N}_{\mathcal{F}_\sigma}\|_1 \lesssim \exp\left(-\frac{\lambda}{\|f\|_\infty}\right) \|f\|_\infty^{-2} \|f\|_2^2.$$

Therefore, assuming for a moment the inclusion (3.8), we have arrived at

$$(3.9) \quad \{|C_{\mathbf{S}}f| \gtrsim \lambda\} \lesssim \exp\left(-\frac{\lambda}{\|f\|_\infty}\right) \|f\|_\infty^{-2} \|f\|_2^2;$$

Proposition 3.1 simply follows from the obvious $\|f\|_\infty^{-1} \|f\|_2 \leq 1$. The above mentioned inclusion is proved by observing that

$$\sup_{x \in E^c} \sup_{\mathbf{T} \in \mathcal{F}_\sigma} |C_{\mathbf{T}}f(x)| \leq \lambda \sigma \log\left(\frac{1}{\sigma}\right),$$

and therefore, making use of the triangle inequality, (3.4), and (3.7),

$$\begin{aligned} |C_{\mathbf{S}}f(x)| &\leq \sum_{\sigma \in 2^{-\mathbb{N}}} |C_{\mathbf{S}_\sigma}f(x)| \leq \sum_{\sigma \in 2^{-\mathbb{N}}} \|\mathcal{W}_{\mathcal{F}_\sigma}\|_\infty \sup_{\mathbf{T} \in \mathcal{F}_\sigma} |C_{\mathbf{T}}f(x)| \\ &\lesssim_\theta \lambda \sum_{\sigma \in 2^{-\mathbb{N}}} \sigma \log\left(\frac{1}{\sigma}\right) \lesssim_\theta \lambda \end{aligned}$$

for $x \in E^c$, which means that $E^c \subset \{C_{\mathbf{S}}f \lesssim \lambda\}$. The proof of Proposition 3.1 is thus completed.

Remark 3.5. Perusing the proof of Proposition 3.1, we realize that we have proved the following estimate: for a finite convex $\mathbf{S} \subset \mathbf{S}_{\mathbb{T}}^{\mathbf{n}}$, and any $A \geq \text{size}_f(\mathbf{S})$,

$$(3.10) \quad |\{x \in \mathbb{T} : |C_{\mathbf{S}}f(x)| \gtrsim \lambda\}| \lesssim_\theta \exp\left(-\frac{\lambda}{A}\right) \frac{\|f\|_p^2}{A^2}, \quad \lambda > 0.$$

This estimate will be used in the proof of Theorem 2.

4. PROOF OF THEOREM 2

By the usual limiting argument, replacing $\mathbf{S}_{\mathbb{T}}^{\mathbf{n}}$ with an arbitrary finite convex subcollection \mathbf{S} , Theorem 2 is equivalent to the estimate

$$(4.1) \quad |\{x \in \mathbb{T} : |C_{\mathbf{S}}f(x)| \gtrsim \log_1(p')\lambda\}| \lesssim_\theta \frac{\|f\|_p^p}{\lambda^p}, \quad \forall \lambda > 0.$$

Furthermore, by scaling f , it suffices to work with $\lambda = 1$.

First of all, note that the left-hand side of (4.1) is less than or equal to

$$(4.2) \quad |\{x \in \mathbb{T} : M_p f(x) > 1\}| + |\{x \in \mathbb{T} : |C_{\mathbf{S}}f(x)| \gtrsim \log_1(p'), M_p f(x) \leq 1\}|$$

and the first summand complies with the bound on the right-hand side of (4.1) by the maximal theorem. Thus it suffices to estimate the second summand of (4.2); note that

$$M_p f(x) \leq 1 \implies C_{\mathbf{S}}f(x) = C_{\mathbf{S}^1}f(x), \quad \mathbf{S}^1 = \left\{s \in \mathbf{S} : \inf_{I_s} M_1 f \leq 1\right\},$$

and thus it suffices to estimate

$$(4.3) \quad \begin{aligned} |\{x \in \mathbb{T} : |C_{\mathbf{S}^1} f(x)| \gtrsim \log_1(p')\}| &\leq |\{x \in \mathbb{T} : |C_{\mathbf{S}^1} f_1(x)| \gtrsim \log_1(p')\}| \\ &\quad + |\{x \in \mathbb{T} : |C_{\mathbf{S}^1} f_2(x)| \gtrsim \log_1(p')\}|, \end{aligned}$$

where $f_1 := f \mathbf{1}_{\{M_p f \leq 1\}}$, $f_2 := f - f_1$. Our reduction has resulted into

$$(4.4) \quad \text{size}_{f_i}(\mathbf{S}^1) \leq 1, i = 1, 2, \quad \|f_1\|_2^2 \leq \|f_1\|_p^p \|f_1\|_\infty^{2-p} \leq \|f\|_p^p,$$

so that the first summand in (4.3) is bounded by invoking estimate (3.10) with $A = 1$:

$$|\{|C_{\mathbf{S}^1} f_1| \gtrsim \log_1(p')\}| \leq |\{|C_{\mathbf{S}^1} f_1| \gtrsim 1\}| \lesssim \|f_1\|_2^2 \leq \|f\|_p^p$$

We are only left with estimating the second summand in (4.3). To do this, our plan is to apply (3.10) again, once we have at hand the following multi-frequency projection lemma, which relies on the structure imposed on $\mathbf{S}_{\mathbb{T}}^{\mathbf{n}}$ by the lacunary sequence \mathbf{n} . The first multi-frequency decomposition lemma of this sort appeared in [15] for the Fourier case, and modified Walsh versions of it have been successfully used in getting uniform estimates [16] and endpoint bounds [5] for the quartile operator. An argument along the same lines, but in the case of multiple lacunary frequencies, appears in [6]: our lemma is an L^p , $1 < p < 2$ reformulation of that argument.

Lemma 4.1. *There is a function $g : \mathbb{T} \rightarrow \mathbb{C}$ with*

$$C_{\mathbf{S}^1} f_2 = C_{\mathbf{S}^1} g, \quad \|g\|_2^2 \lesssim (p')^2 |\{M_p f > 1\}|$$

Indeed, in view of Lemma 4.1, a further application of (3.10) with $A = 1$ yields

$$|\{|C_{\mathbf{S}^1} f_2| \gtrsim \log_1(p')\}| = |\{|C_{\mathbf{S}^1} g| \gtrsim \log_1(p')\}| \lesssim e^{-2\log_1(p')} \|g\|_2^2 \lesssim |\{M_p f > 1\}|,$$

which once again has the correct measure by the maximal theorem. We have completed the proof of Theorem 2, up to showing Lemma 4.1.

Proof of Lemma 4.1. Let $I \in \mathbf{I}$ be the maximal dyadic intervals of $\{M_p f_1 > 1\}$; for each $I \in \mathbf{I}$, let $t \in T_I$ be the collection of all tiles having $I_t = I$ and which are comparable under \ll to some tile in $\{s_1 : s \in \mathbf{S}^1\}$. These are obviously pairwise disjoint. The definition of \mathbf{S}^1 ensures that whenever $I_s \cap I$ for some $s \in \mathbf{S}_1$ and $I \in \mathbf{I}$, it must be that $I \subsetneq I_s$. It follows that if $t \in T_I, s_1 \in \{s_1 : s \in \mathbf{S}^1\}$ are related, then $t \ll s_1, s_2$. In particular, each $t \in T_I$ must contain some lacunary frequency $n_j \in \mathbf{n}$; furthermore, by standard properties of Walsh wave packets, w_{s_1} (and w_{s_2} as well, but we will not need this) is a scalar multiple of w_t on I , and, in particular, $w_{s_1} \mathbf{1}_I$ belongs to H_I , the subspace of $L^2(I)$ spanned by $\{w_t : t \in T_I\}$. For functions $v \in H_I$, one has the estimate

$$(4.5) \quad \|v\|_{L^q(I)} \lesssim q \|v\|_{\text{BMO}(I)} \lesssim_\theta q \|v\|_{L^2(I)}, \quad 2 < q < \infty;$$

the first bound is simply John-Nirenberg's inequality (and $\text{BMO}(I)$ is the dyadic version), while the second is proved in [9]. Since $\|f_2\|_{L^p(I)} = \|f\|_{L^p(I)} \leq 2$ by maximality of I in $\{M_p f > 1\}$, it then follows that

$$|(f_2, v)_{L^2(I)}| \leq \|f_2\|_{L^p(I)} \|v\|_{L^{p'}(I)} \lesssim_\theta p' \|v\|_{L^2(I)} \quad \forall v \in H_I.$$

Therefore g_I , the projection of $f_2 \mathbf{1}_I$ on H_I , satisfies $\|g_I\|_{L^2(I)} \lesssim p'$, and defining $g := \sum_{I \in \mathbf{I}} g_I$, we see that

$$\|g\|_2^2 = \sum_{I \in \mathbf{I}} |I| \|g_I\|_{L^2(I)}^2 \lesssim_\theta (p')^2 \sum_{I \in \mathbf{I}} |I| = (p')^2 |\{M_p f > 1\}|$$

Finally, in view of the above discussion, if $s_1 \in \{s_1 : s \in \mathbf{S}^1\}$

$$\langle f_2, w_{s_1} \rangle = \sum_{I \in \mathbf{I}} \langle f_2, w_{s_1} \mathbf{1}_I \rangle = \sum_{I \in \mathbf{I}} \langle f_2 \mathbf{1}_I, c w_{t(s_1)} \rangle = \sum_{I \in \mathbf{I}} \langle g_I, w_{s_1} \rangle = \langle g, w_{s_1} \rangle$$

where $t(s_1)$ is the unique (if any) element t of T_I with $t \ll s_1$. This proves that $C_{\mathbf{S}^1} f_2 = C_{\mathbf{S}^1} g$, and finishes the proof of the lemma. \square

5. REMARKS AND COMPLEMENTS

5.1. A comparison with the argument in [6]. Therein, estimate (1.2) follows by upgrading the restricted weak-type version (1.1), via Antonov's lemma [1, 17] (which uses the structure of the Walsh-Carleson kernel). In turn, (1.1) is a consequence of the restricted weak-type estimate

$$(5.1) \quad \langle C^{\mathbf{n}} f, g \rangle \lesssim |F| \log_2 \left(\frac{|G|}{|F|} \right)$$

for all sets $F, G \subset \mathbb{T}$, and all functions $|f| \leq \mathbf{1}_F$, $|g| \leq \mathbf{1}_{G'}$, with G' being a suitably chosen major subset of G . The proof of (5.1) follows the usual Lacey-Thiele argument for boundedness of the unrestricted Carleson operator [12]; in particular, the dual quantity (density)

$$\text{dense}(\mathbf{S}) = \sup_{s \in \mathbf{S}} \frac{|I_s \cap N^{-1}(\omega_s) \cap G|}{|I_s|}$$

comes into play. For the unrestricted Carleson operator, the analogue of (5.1) holds with a single logarithm; the improvement to double logarithm is possible thanks to a multifrequency projection argument based on the same tools as Lemma 4.1 (in particular, an improvement over Hausdorff-Young inequality in the vein of (4.5)).

Our proof of Theorem 1 yields (1.2) directly from the weak L^p estimate

$$(5.2) \quad B_{\mathbf{n}}(p) := \|C^{\mathbf{n}} f\|_{L^p(\mathbb{T}) \rightarrow L^{p,\infty}(\mathbb{T})} \lesssim_{\theta} \log_1(p'), \quad \forall 1 < p < 2$$

of Theorem 2, avoiding the need for extrapolation techniques. Moreover, our arguments do not employ density (which is also the key quantity in the proof of the Fourier case [13]), relying instead on the property that any collection of bitiles $\mathbf{S} \subset \mathbf{S}_{\mathbb{T}}^{\mathbf{n}}$ can be arranged into a forest \mathcal{F} of trees with

$$(5.3) \quad |C_{\mathbf{S}} f(x)| \lesssim_{\theta} \sup_{\mathbf{T} \in \mathcal{F}} |C_{\mathbf{T}} f(x)|,$$

which exploits the lacunary structure, see Lemma 3.4. This property reflects the fact that the lacunary Carleson operator is essentially a supremum of (lacunarily) modulated Hilbert transforms acting on (essentially) pairwise disjoint regions of the time-frequency plane.

5.2. Sharpness of Theorem 2. We conjecture that Theorem 2, summarized into (5.2), is sharp in the following sense: for a generic lacunary sequence,

$$\limsup_{p \rightarrow 1^+} \frac{B_{\mathbf{n}}(p)}{\varphi(p')} = \infty \quad \forall \varphi(t) = o(\log_1(t)), \quad t \rightarrow \infty.$$

We cannot quite prove this result; however, the weaker statement

$$\limsup_{p \rightarrow 1^+} \frac{B_{\mathbf{n}}(p)}{\varphi(p')} = \infty \quad \forall \varphi(t) = o\left(\frac{\log_1(t)}{\log_3(t)}\right), \quad t \rightarrow \infty.$$

must hold. If it were not so, an argument along the lines of the proof of Theorem 1 would contradict Konyagin's counterexample from [10], that we have mentioned at the beginning of the paper. Similarly, proving that $B_{\mathbf{n}}(p) \sim_{\theta} O(\log_1(p')/\log_3(p'))$ would allow the removal of the quadruple-log term in Theorem 1, thus yielding the sharp result. Our conjecture stems from deeming the term $\log_3(p')$ as inconsequential, and expresses the belief that knowing the sharp weak L^p constant would not suffice to prove the sharp analogue of Theorem 1.

5.3. Strong L^1 bounds. A further unresolved question concerns the largest Orlicz space X of functions $\mathbb{T} \rightarrow \mathbb{C}$ for which the bound

$$\|W_{\mathbf{n}}^* f\|_{L^1(\mathbb{T})} \lesssim_{\theta} \|f\|_X$$

holds. Since $W_{\mathbf{n}}^*$ is greater than each (discrete) n_j -modulated Hilbert transform, it follows that no Orlicz space $L_{\varphi}(\mathbb{T})$ with

$$\limsup_{t \rightarrow \infty} \frac{\varphi(t)}{t \log_1(t)} = \infty$$

embeds into X . The (sharp, in terms of Orlicz norms) inclusion $L \log L(\mathbb{T}) \subset X$ is still unknown: the current best result [6, (1.6) of Theorem 1.4] is that $L \log_1 L \log_2 L(\mathbb{T}) \subset X$. We can easily recover this result from Theorem 2: applying Marcinkiewicz interpolation, one turns the weak-type bound of Theorem 2 into the strong bound

$$\|W_{\mathbf{n}}^*\|_{p \rightarrow p} \lesssim_{\theta} p' \log_1(p'),$$

which in turn implies $W_{\mathbf{n}}^* : L \log_1 L \log_2 L(\mathbb{T}) \rightarrow L^1(\mathbb{T})$, repeating the proof of the classical Yano extrapolation theorem.

In relation to this, it is known that all sublinear translation invariant operators of restricted weak type (1,1) (of which $W_{\mathbf{n}}^*$ is an instance) map $L \log_1 L(\mathbb{T})$ into $L^1(\mathbb{T})$ (see for example [7]). However, a result of Moon [14] implies that an operator of the form $Tf = \sup_n |f * g_n|$ with each $g_n \in L^1(\mathbb{T})$, is of restricted weak type (1,1) if and only if it is of weak type (1,1). Since $W_{\mathbf{n}}^*$ is of this form, and it is not weak type (1,1), it cannot be restricted weak type (1,1) either. This suggests the need for direct methods in the search for a proof that $W_{\mathbf{n}}^*$ is strong-type $L \log_1 L(\mathbb{T}) \rightarrow L^1(\mathbb{T})$, possibly relying on (5.3).

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